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On the Bohr inequality of operators [☆]

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Abstract

This paper is focused on the operator inequalities of the Bohr type. We will give a new and transparent proof for the operator Bohr inequality through an absolute value operator identity, show some related operator inequalities by means of 2×2 (block) operator matrices, and finally we will present a generalization of the operator Bohr inequality for multiple operators.

Keywords: Bohr inequality; Matrix absolute value; Matrix inequality; Modulus; Operator inequality; Operator absolute value; Polar decomposition; Positive operator

1. Introduction

Let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on a complex separable Hilbert space H. As usual, I is the identity operator and $\mathbb{R}$ is the set of real numbers. Denote by $|A|$ the absolute value operator (or modulus) of $A \in \mathcal{B}(H)$:

$$|A| = (A^*A)^{1/2},$$

where A^* is the adjoint operator of A. Note that $|A| = 0$ if and only if $A = 0$.

We write $A \geqslant 0$ if A is a positive operator, meaning $(Ax, x) \geqslant 0$ for all $x \in H$, and $A \geqslant B$ if A and B are self-adjoint operators and if $A - B \geqslant 0$.

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There exist a great number of results on the absolute value operator. As an example, in the case of finite dimension, for any $n \times n$ square complex matrices A and B, there are unitary matrices U and V depending on A and B such that

$$|A + B| \leqslant U^*|A|U + V^*|B|V,$$

where the presence of U and V is necessary. This is so-called Thompson matrix triangle inequality [4] (or [7, p. 237]). An operator version of the triangle inequality is discussed in [1]. The absolute value operator is of fundamental importance since it is the positive part in the polar decomposition $A = U|A|$.

This paper is focused on the operator inequality of the Bohr type. Operator matrices will serve as a basic tool. Let $A, B, C, D \in \mathcal{B}(\mathcal{H})$. Then the operator matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is regarded as an operator on the direct sum $H \oplus H$, in which elements are thought of as column vectors, and (A, B) and $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ are operators from $H \oplus H$ to H and H to $H \oplus H$, respectively. (See, for instance, [6, p. 145].)

2. Operator Bohr inequality via identities

The classical Bohr inequality (see, e.g., [3, p. 312]) for scalars asserts that for complex numbers a, b and real numbers $p, q > 1$ such that $1/p + 1/q = 1$,

$$|a - b|^2 \leqslant p|a|^2 + q|b|^2. \tag{1}$$

An operator version of the Bohr inequality is obtained by Hirzallah [2].

Theorem 1. [2] *Let $A, B \in \mathcal{B}(H)$, $p, q > 1$, $1/p + 1/q = 1$, $p \leqslant q$. Then*

$$|A - B|^2 + \left|(1 - p)A - B\right|^2 \leqslant p|A|^2 + q|B|^2. \tag{2}$$

Corollary 1. [2] *Let $A, B \in \mathcal{B}(H)$, $p, q > 1$, $1/p + 1/q = 1$. Then*

$$|A - B|^2 \leqslant p|A|^2 + q|B|^2. \tag{3}$$

Equality holds if and only if $B = (1 - p)A$.

Note that in (the proof of) Hirzallah's theorem, the condition $p \leqslant q$, which implies $1 < p \leqslant 2$ and $q \geqslant 2$, is necessary. We now present an operator identity from which (2) follows immediately and the condition $p \leqslant q$ is removed.

Theorem 2. *Let $A, B \in \mathcal{B}(H)$, $p, q > 1$, $1/p + 1/q = 1$. Then*

$$|A - B|^2 + \left|\sqrt{p/q}A + \sqrt{q/p}B\right|^2 = p|A|^2 + q|B|^2. \tag{4}$$

Equivalently, for any α, $0 \leqslant \alpha \leqslant 1$,

$$\left|\alpha A + (1 - \alpha)B\right|^2 + \alpha(1 - \alpha)|A - B|^2 = \alpha|A|^2 + (1 - \alpha)|B|^2. \tag{5}$$

Proof. Expanding

$$|A - B|^2 = |A|^2 + |B|^2 - (A^*B + B^*A)$$

and

$$\left|\sqrt{p/q}A + \sqrt{q/p}B\right|^2 = p/q|A|^2 + q/p|B|^2 + (A^*B + B^*A)$$

and adding these identities, we get

$$|A-B|^2+\left|\sqrt{p/q}A+\sqrt{q/p}B\right|^2=(1+p/q)|A|^2+(1+q/p)|B|^2.$$

This is the same as (4), since $1/p+1/q=1$ yields $1+p/q=p$, $1+q/p=q$.

To see that identity (4) is equivalent to (5), we divide both sides of (4) by pq. Then setting $\alpha=1/q$ will reveal identity (5). □

Note that in (4), $\sqrt{p/q}$ and $\sqrt{q/p}$ can be replaced by $\sqrt{p-1}$ and $\sqrt{q-1}$, respectively. Identity (5) gives immediately the square-convexity inequality

$$\left|\alpha A+(1-\alpha)B\right|^2\leqslant\alpha|A|^2+(1-\alpha)|B|^2 \tag{6}$$

which is essentially the same as (3).

Inequality (3) follows from (4) at once. For the equality case, note that $\sqrt{p/q}A+\sqrt{q/p}B=0$ if and only if $p/qA+B=0$, i.e., $B=(1-p)A$.

To see that (2) follows from (4) all we need to show is when $1\leqslant p\leqslant 2$,

$$\left|(1-p)A-B\right|^2\leqslant\left|\sqrt{p/q}A+\sqrt{q/p}B\right|^2. \tag{7}$$

Instead of showing this particular inequality, we consider inequalities in more general form with real parameters x, y, s, t,

$$|xA+yB|^2\leqslant|sA+tB|^2.$$

For this purpose, we show a lemma that will be repeatedly used later. The result is of interest in its own right.

Lemma 1. *Let $A, B\in\mathcal{B}(H)$. If $a, b>0$, $c\in\mathbb{R}$, and $ab\geqslant c^2$, then*

$$a|A|^2+b|B|^2+c(A^*B+B^*A)\geqslant 0. \tag{8}$$

Proof. Since

$$\begin{pmatrix} a & c\\ c & b\end{pmatrix}\geqslant 0,\qquad \begin{pmatrix} A^*\\ B^*\end{pmatrix}(A,B)=\begin{pmatrix} |A|^2 & A^*B\\ B^*A & |B|^2\end{pmatrix}\geqslant 0,$$

we have

$$\begin{pmatrix} a|A|^2 & cA^*B\\ cB^*A & b|B|^2\end{pmatrix}\geqslant 0.$$

Thus

$$(I,I)\begin{pmatrix} a|A|^2 & cA^*B\\ cB^*A & b|B|^2\end{pmatrix}\begin{pmatrix} I\\ I\end{pmatrix}=a|A|^2+b|B|^2+c(A^*B+B^*A)\geqslant 0. \qquad \square$$

Lemma 2. *Let $A, B\in\mathcal{B}(H)$. If $x, y, s, t\in\mathbb{R}$ such that*

$$|x|\leqslant|s|,\quad |y|\leqslant|t|,\quad xt=sy,$$

then

$$|xA+yB|^2\leqslant|sA+tB|^2. \tag{9}$$

Proof. This can be proven by expanding both sides of (9), moving the terms on the left-hand side to the right-hand side, then making use of Lemma 1 with

$$a = s^2 - x^2, \quad b = t^2 - y^2, \quad c = st - xy. \qquad \square$$

If $1 \leqslant p \leqslant q$, then $1 \leqslant p \leqslant 2$. Taking $x = p - 1$, $y = 1$, $s = \sqrt{p/q}$, $t = \sqrt{q/p}$ in (9), inequality (7) and thus (2) follow at once. One can also obtain (6) from Lemma 1 directly by rewriting (6) as (8) with $a = b = c = \alpha(1 - \alpha)$.

Theorem 2 can generate a variety of inequalities similar to (2). In fact, putting $s = \sqrt{p/q}$ and $t = \sqrt{q/p}$ in Lemma 2, for any real numbers x and y satisfying $xq = yp$ and $x^2 \leqslant p/q = p - 1$, we have

$$|A - B|^2 + |xA + yB|^2 \leqslant p|A|^2 + q|B|^2. \tag{10}$$

In particular, when $1 \leqslant p \leqslant q$ and $1/p + 1/q = 1$, setting $x = (p - 1)^k$ and $y = (p - 1)^{k-1}$, where k is any positive integer, we arrive at

$$|A - B|^2 + \left|(p - 1)^k A + (p - 1)^{k-1} B\right|^2 \leqslant p|A|^2 + q|B|^2 \tag{11}$$

which reduces to (2) when $k = 1$. The term $|A - B|^2$ in (10) will be replaced with a more general term $|\alpha A + \beta B|^2$ in the next section (see Theorem 6).

Returning to Lemma 1, letting $a = p - 1$, $b = 1/(p - 1)$, $c = \pm 1$, we have

Corollary 2. [2] *Let $A, B \in \mathcal{B}(H)$. Then for any $p > 1$,*

$$\pm(A^*B + B^*A) \leqslant (p - 1)|A|^2 + \frac{1}{p - 1}|B|^2.$$

It follows that, by setting $p = 2$,

$$\pm(A^*B + B^*A) \leqslant |A|^2 + |B|^2, \tag{12}$$

which may be compared in parallel to the matrix Hadamard product inequality [5, Corollary 12]:

$$\pm(A^*B \circ B^*A) \leqslant |A|^2 \circ |B|^2.$$

Note that $\pm(A^*B + B^*A) \leqslant |A^*B + B^*A| \nleqslant |A^*B| + |B^*A|$ in general.

The following theorem sharpens the inequality in Corollary 2.

Theorem 3. *Let $A, B \in \mathcal{B}(H)$. Then for any real number $t \neq 0$,*

$$\pm(A^*B + B^*A) \leqslant \frac{1}{2}|tA \pm 1/tB|^2 \leqslant t^2|A|^2 + 1/t^2|B|^2.$$

Proof. Since

$$|tA + 1/tB|^2 = t^2|A|^2 + 1/t^2|B|^2 + (A^*B + B^*A)$$

and

$$|tA - 1/tB|^2 = t^2|A|^2 + 1/t^2|B|^2 - (A^*B + B^*A),$$

we have

$$|tA + 1/tB|^2 + |tA - 1/tB|^2 = 2\big(t^2|A|^2 + 1/t^2|B|^2\big)$$

and

$$|tA + 1/tB|^2 - |tA - 1/tB|^2 = 2(A^*B + B^*A).$$

So

$$2(A^*B + B^*A) \leqslant |tA + 1/tB|^2 \leqslant 2\big(t^2|A|^2 + 1/t^2|B|^2\big)$$

or

$$A^*B + B^*A \leqslant \frac{1}{2}|tA + 1/tB|^2 \leqslant t^2|A|^2 + 1/t^2|B|^2.$$

The other inequality with negative sign is similarly proven. □

By putting $t = 1$, we get

$$\pm(A^*B + B^*A) \leqslant \frac{1}{2}|A \pm B|^2 \leqslant |A|^2 + |B|^2.$$

We end this section by noting that more operator identities can be shown in a similar way as in Theorem 2. We present two more inequalities below that may be of interest to generate some related inequalities.

Theorem 4. *Let $A, B \in \mathcal{B}(H)$. Then for any $\alpha, \beta \in \mathbb{R}$,*

$$|\alpha A + B|^2 + |A + \beta B|^2 = \big(1 + \alpha^2\big)|A|^2 + \big(1 + \beta^2\big)|B|^2 + (\alpha + \beta)(A^*B + B^*A).$$

Theorem 5. *Let $A, B \in \mathcal{B}(H)$ and $\alpha, \beta \in \mathbb{R}$. If $\alpha + \beta + \alpha\beta = 0$, then*

$$|\alpha A + B|^2 + |A + \beta B|^2 + |\alpha A + \beta B|^2 = \big(2\alpha^2 + 1\big)|A|^2 + \big(2\beta^2 + 1\big)|B|^2.$$

3. More inequalities via 2 × 2 block matrices

In this section we present more inequalities resembling (2) through operator matrices. Our purpose is to compare $|A + B|^2$ to $|A|^2$ and $|B|^2$.

First observe that

$$|A + B|^2 = (I, I)\begin{pmatrix} |A|^2 & A^*B \\ B^*A & |B|^2 \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix} \geqslant 0.$$

Thus, we can associate each absolute value square of the sum of two operators with a 2×2 block operator matrix. Writing in symbols, we have

$$|A + B|^2 \mapsto \begin{pmatrix} |A|^2 & A^*B \\ B^*A & |B|^2 \end{pmatrix}.$$

And this map is addition-preservative. Furthermore, if for $A, B, C, D \in \mathcal{B}(H)$,

$$\begin{pmatrix} |A|^2 & A^*B \\ B^*A & |B|^2 \end{pmatrix} \leqslant \begin{pmatrix} |C|^2 & C^*D \\ D^*C & |D|^2 \end{pmatrix},$$

then

$$|A + B|^2 \leqslant |C + D|^2.$$

This suggests that one may convert a problem of absolute value operators to a problem of 2×2 operator matrices. On many circumstances the later approach is more transparent and easy to handle. Consider, for instance, the inequality

$$|\alpha A + \beta B|^2 \leqslant x|A|^2 + y|B|^2, \quad \alpha, \beta \in \mathbb{R}, \ x, y > 0. \tag{13}$$

Since

$$|\alpha A + \beta B|^2 \mapsto \begin{pmatrix} \alpha^2|A|^2 & \alpha\beta A^*B \\ \alpha\beta B^*A & \beta^2|B|^2 \end{pmatrix}$$

and

$$x|A|^2 + y|B|^2 \mapsto \begin{pmatrix} x|A|^2 & 0 \\ 0 & y|B|^2 \end{pmatrix},$$

thus if

$$\begin{pmatrix} \alpha^2|A|^2 & \alpha\beta A^*B \\ \alpha\beta B^*A & \beta^2|B|^2 \end{pmatrix} \leqslant \begin{pmatrix} x|A|^2 & 0 \\ 0 & y|B|^2 \end{pmatrix}$$

then the inequality (13) holds. This leads to the condition for (13) to hold:

$$\left(x - \alpha^2\right)\left(y - \beta^2\right) \geqslant \alpha^2\beta^2$$

or equivalently

$$xy \geqslant x\beta^2 + y\alpha^2.$$

Theorem 6. *Let $A, B \in \mathcal{B}(H)$, $\alpha, \beta, u, v \in \mathbb{R}$, $p, q > 0$. If*

$$p \geqslant \alpha^2 + u^2, \qquad q \geqslant \beta^2 + v^2$$

and

$$\left[p - \left(\alpha^2 + u^2\right)\right]\left[q - \left(\beta^2 + v^2\right)\right] \geqslant (\alpha\beta + uv)^2,$$

then

$$|\alpha A + \beta B|^2 + |uA + vB|^2 \leqslant p|A|^2 + q|B|^2.$$

Proof. Notice that

$$|\alpha A + \beta B|^2 \mapsto \begin{pmatrix} \alpha^2|A|^2 & \alpha\beta A^*B \\ \alpha\beta B^*A & \beta^2|B|^2 \end{pmatrix},$$

$$|uA + vB|^2 \mapsto \begin{pmatrix} u^2|A|^2 & uvA^*B \\ uvB^*A & v^2|B|^2 \end{pmatrix}.$$

Adding them gives

$$|\alpha A + \beta B|^2 + |uA + vB|^2 \mapsto \begin{pmatrix} (\alpha^2 + u^2)|A|^2 & (\alpha\beta + uv)A^*B \\ (\alpha\beta + uv)B^*A & (\beta^2 + v^2)|B|^2 \end{pmatrix}$$

which is to be dominated ($\leqslant$) by

$$p|A|^2 + q|B|^2 \mapsto \begin{pmatrix} p|A|^2 & 0 \\ 0 & q|B|^2 \end{pmatrix}.$$

This leads to the condition

$$\begin{pmatrix} p-(\alpha^2+u^2) & -(\alpha\beta+uv) \\ -(\alpha\beta+uv) & q-(\beta^2+v^2) \end{pmatrix} \geqslant 0,$$

that is,

$$\left[p-\left(\alpha^2+u^2\right)\right]\left[q-\left(\beta^2+v^2\right)\right] \geqslant (\alpha\beta+uv)^2. \qquad \square$$

We note that inequalities (2) and (3) are immediate from Theorem 6 by taking $\alpha = 1$, $\beta = -1$, $u = 1 - p$, $v = -1$ and $\alpha = 1$, $\beta = -1$, $u = v = 0$, respectively. A variety of inequalities can be obtained by choosing different values of the parameters. Below, for instance, is another one.

Corollary 3. *Let $A, B \in \mathcal{B}(H)$. Then for any $\theta \in \mathbb{R}$, $p, q > 1$, $1/p + 1/q = 1$,*

$$|\sin\theta A + \cos\theta B|^2 + |\cos\theta A + \sin\theta B|^2 \leqslant p|A|^2 + q|B|^2.$$

4. The Bohr inequality for multiple operators

A generalization of the Bohr inequality (1) states that [3, p. 312] for complex numbers $z_1, \ldots, z_k$ and positive numbers $a_1, \ldots, a_k$ such that $\sum_{i=1}^k 1/a_i = 1$,

$$|z_1 + \cdots + z_k|^2 \leqslant a_1|z_1|^2 + \cdots + a_k|z_k|^2.$$

Or equivalently, putting in convexity, for positive $t_1, \ldots, t_k$ with $\sum_{i=1}^k t_i = 1$,

$$|t_1 z_1 + \cdots + t_k z_k|^2 \leqslant t_1|z_1|^2 + \cdots + t_k|z_k|^2.$$

We now present an analog of this inequality for operators.

Theorem 7. *Let k be a positive integer and let $A_i \in \mathcal{B}(H)$, $i = 1, \ldots, k$. Then for any set of positive numbers $t_1, \ldots, t_k$ such that $\sum_{i=1}^k t_i = 1$,*

$$|t_1 A_1 + \cdots + t_k A_k|^2 \leqslant t_1|A_1|^2 + \cdots + t_k|A_k|^2.$$

Proof. We use mathematical induction on k. When $k = 2$, the inequality holds as discussed in Section 2. Suppose that the inequality holds for $k - 1$, $k > 2$.

Let $B = t_1/(1-t_k)A_1 + \cdots + t_{k-1}/(1-t_k)A_{k-1}$. Since

$$t_1/(1-t_k) + \cdots + t_{k-1}/(1-t_k) = 1,$$

it follows that

$$|B|^2 \leqslant t_1/(1-t_k)|A_1|^2 + \cdots + t_{k-1}/(1-t_k)|A_{k-1}|^2.$$

Thus,

$$(1-t_k)|B|^2 \leqslant t_1|A_1|^2 + \cdots + t_{k-1}|A_{k-1}|^2.$$

So we have

$$\begin{aligned} |t_1 A_1 + \cdots + t_k A_k|^2 &= \left|(1-t_k)B + t_k A\right|^2 \\ &\leqslant (1-t_k)|B|^2 + t_k|A|^2 \\ &\leqslant t_1|A_1|^2 + \cdots + t_k|A_k|^2. \qquad \square \end{aligned}$$

We end the paper by pointing out that the aforementioned inequalities without squares such as (see (6), for instance)

$$\left|\alpha A + (1-\alpha)B\right| \leqslant \alpha|A| + (1-\alpha)|B|$$

are appealing but may be invalid in general. In particular the inequality

$$\left|\alpha A + (1-\alpha)A^*\right| \leqslant \alpha|A| + (1-\alpha)|A^*|$$

does not hold unless A is normal, as one may check the counterexample:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \alpha = 1/2.$$

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